

Multibump nodal solutions for an indefinite superlinear elliptic problem*

Pedro M. Girão[†] and José Maria Gomes[‡]

Instituto Superior Técnico

Av. Rovisco Pais

1049-001 Lisbon, Portugal

Abstract

We define some Nehari-type constraints using an orthogonal decomposition of the Sobolev space H_0^1 and prove the existence of multibump nodal solutions for an indefinite superlinear elliptic problem.

1 Introduction

Consider a Lipschitz bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, and a function $a \in C(\bar{\Omega})$, with $a = a^+ - a^-$, where $a^+ = \max\{a, 0\}$ as usual. Assume the set $a^+ > 0$ is the union of a finite number, $L \geq 1$, of open connected and disjoint Lipschitz components. We separate the components arbitrarily into three families

$$\begin{aligned}\Omega^+ = \{x \in \Omega : a^+(x) > 0\} &= (\cup_{i=1}^I \tilde{\omega}_i) \cup (\cup_{j=1}^J \hat{\omega}_j) \cup (\cup_{k=1}^K \bar{\omega}_k) \\ &= \tilde{\Omega} \cup \hat{\Omega} \cup \underline{\Omega},\end{aligned}$$

so that $L = I + J + K$; we also assume

$$\Omega^- = \{x \in \Omega : a^-(x) > 0\} = \Omega \setminus \overline{\Omega^+}.$$

Let $\mu > 0$ and p be a superquadratic and subcritical exponent, $2 < p < 2^*$, with $2^* = 2N/(N-2)$ for $N \geq 3$, and $2^* = +\infty$ for $N = 1$ or 2 . Our main result is

Theorem 1.1. *For every large μ , there exists an $H_0^1(\Omega)$ weak solution u_μ of*

$$-\Delta u = (a^+ - \mu a^-)|u|^{p-2}u \quad \text{in } \Omega. \quad (1)$$

*2000 Mathematics Subject Classification: 35J65 (35J20)

Keywords: Multibump solutions, Nehari manifold, sign-changing solutions, elliptic equations

[†]Email: pgirao@math.ist.utl.pt. Partially supported by the Center for Mathematical Analysis, Geometry and Dynamical Systems through FCT Program POCTI/FEDER and by grant POCI/FEDER/MAT/55745/2004.

[‡]Email: jgomes@math.ist.utl.pt. Supported by FCT grant SFRH/BPD/29098/2006.

Furthermore, the family $\{u_\mu\}$ has the property that (modulo a subsequence)

$$u_\mu \rightharpoonup u \quad \text{in } H_0^1(\Omega) \text{ as } \mu \rightarrow +\infty, \quad (2)$$

where

$$\begin{cases} -\Delta u = a^+|u|^{p-2}u & \text{in } \tilde{\omega}_i, \\ u^\pm \not\equiv 0 & \text{in } \tilde{\omega}_i, \end{cases} \quad i = 1, \dots, I,$$

$$\begin{cases} -\Delta u = a^+|u|^{p-2}u & \text{in } \hat{\omega}_j, \\ u^+ \not\equiv 0, \quad u^- \equiv 0 & \text{in } \hat{\omega}_j, \end{cases} \quad j = 1, \dots, J,$$

$$u \equiv 0 \quad \text{in } \bar{\omega}_k, \quad k = 1, \dots, K,$$

and

$$u \equiv 0 \quad \text{in } \Omega^-.$$

The one-dimensional version of (1) was studied in [15] with topological shooting arguments and phase-plane analysis. Theorem 1.1 extends the main result in [7] where the case $\tilde{\Omega} = \emptyset$ was considered, so that the function u in (2) was positive. The authors used a volume constrain regarding the L^p norm, rescaling and a min-max argument based on the Mountain Pass Lemma. A careful analysis allowed them to distinguish between the solutions that arise from the 2^L different possible partitionings of $\Omega^+ = \hat{\Omega} \cup \underline{\Omega}$. However, the argument in [7] does not seem either to extend easily to the present situation or to be suited to non-homogeneous nonlinearities.

Our approach is adapted from the work [18] regarding a system of equations related to

$$\begin{cases} -\epsilon^2 \Delta u + V(x)u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

when ϵ is small and the functions V and f satisfy appropriate conditions. The positive function V was assumed to have a finite number of minima. In particular, the authors proved the existence of multipeak positive solutions by defining a Nehari-type manifold which, roughly speaking, imposes that the derivative of the associated Euler-Lagrange functional at a function u should vanish when applied to a truncation of u around a minimum of the potential function V .

The perspective of [18] is related to the one of [16] which, using Nehari conditions and a cut-off operator, simplifies the original techniques for gluing together mountain-pass type solutions of [12], [13] and [20].

Our method consists in defining a Nehari-type set, \mathcal{N}_μ , by imposing that the derivative of the associated Euler-Lagrange functional at a function u should vanish when applied to the positive and negative parts of some projections of u . The idea to use these projections is borrowed from [7], where they are also used, but in a different way.

We prove that the Euler-Lagrange functional associated to (1) has a minimum over the set \mathcal{N}_μ using an argument similar to the one found in [8]. Since our set \mathcal{N}_μ is not a manifold (see [5, Lemma 3.1]), one has to demonstrate, as in [9], that the minima are indeed critical points. As mentioned above, in the case that $\tilde{\Omega} = \emptyset$ we recover the main result of [7], but with a simpler proof.

Our results are somewhat parallel to the ones of singular perturbation problems like in [14]. The large parameter μ in (1) plays the role of the small parameter ϵ . The solutions concentrate in the set $\tilde{\Omega} \cup \hat{\Omega}$ and vanish in the set $\underline{\Omega} \cup \Omega^-$ as $\mu \rightarrow +\infty$.

In [1] flow invariance properties together with a weak splitting condition proved the existence of infinitely many geometrically distinct two bump solutions of a periodic superlinear Schrödinger equation. The paper [4] is concerned with the singular perturbed equation above. As a special case, the authors observed the existence of multiple pairs of concentrating nodal solutions at an isolated minimum of the potential.

There has been much interest in elliptic problems with a sign changing weight. We refer to [2], [3], [6], [11], [17], [19], [21] and the references therein.

For simplicity we restrict the proof to the case where $I = J = K = 1$, but it extends to the other ones as well. The work is organized as follows. In Section 2 we provide estimates for minimizing sequences on the set \mathcal{N}_μ . In Section 3 we prove the existence of a minimizer in the set \mathcal{N}_μ . Finally, in Section 4 we prove that a minimizer in the set \mathcal{N}_μ is a critical point using a local deformation and a degree argument similar to the one in [10].

2 Estimates for minimizing sequences on a Nehari-type set \mathcal{N}_μ

As mentioned in the Introduction, we consider a Lipschitz bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, and a function $a \in C(\bar{\Omega})$. We assume the set $a^+ > 0$ is the union of three Lipschitz components,

$$\{x \in \Omega : a^+(x) > 0\} = \tilde{\omega} \cup \hat{\omega} \cup \bar{\omega},$$

and

$$\{x \in \Omega : a^-(x) > 0\} = \Omega \setminus \overline{(\tilde{\omega} \cup \hat{\omega} \cup \bar{\omega})}. \quad (3)$$

We introduce a positive parameter μ and consider $2 < p < 2^*$.

We denote by $\langle \cdot, \cdot \rangle$ the usual inner product on the Sobolev space $H_0^1(\Omega)$, i.e. $\langle u, v \rangle = \int \nabla u \cdot \nabla v$ for $u, v \in H_0^1(\Omega)$. When the region of integration is

not specified it is understood that the integrals are over Ω . We denote by $\| \cdot \|$ the induced norm. We define the spaces

$$\begin{aligned}\underline{H}(\tilde{\omega}) &= \{u \in H_0^1(\Omega) : u = 0 \text{ in } \Omega \setminus \tilde{\omega}\}, \\ \underline{H}(\hat{\omega}) &= \{u \in H_0^1(\Omega) : u = 0 \text{ in } \Omega \setminus \hat{\omega}\}, \\ \underline{H}(\bar{\omega}) &= \{u \in H_0^1(\Omega) : u = 0 \text{ in } \Omega \setminus \bar{\omega}\},\end{aligned}$$

which can be obtained from the spaces $H_0^1(\tilde{\omega})$, $H_0^1(\hat{\omega})$, $H_0^1(\bar{\omega})$ by extending functions as zero on $\Omega \setminus \tilde{\omega}$, $\Omega \setminus \hat{\omega}$, $\Omega \setminus \bar{\omega}$, respectively.

Each $u \in H_0^1(\Omega)$ can be decomposed as

$$u = \tilde{u} + \hat{u} + \bar{u} + \underline{u},$$

with \tilde{u} , \hat{u} and \bar{u} the projections of u on $\underline{H}(\tilde{\omega})$, $\underline{H}(\hat{\omega})$ and $\underline{H}(\bar{\omega})$, respectively. We recall the projections are defined by

$$\begin{aligned}\tilde{u} \in \underline{H}(\tilde{\omega}) : \forall \varphi \in \underline{H}(\tilde{\omega}), \quad \langle u, \varphi \rangle &= \langle \tilde{u}, \varphi \rangle, \\ \hat{u} \in \underline{H}(\hat{\omega}) : \forall \varphi \in \underline{H}(\hat{\omega}), \quad \langle u, \varphi \rangle &= \langle \hat{u}, \varphi \rangle, \\ \bar{u} \in \underline{H}(\bar{\omega}) : \forall \varphi \in \underline{H}(\bar{\omega}), \quad \langle u, \varphi \rangle &= \langle \bar{u}, \varphi \rangle.\end{aligned}$$

Clearly, these projections are orthogonal and continuous with respect to the weak topology. The function \underline{u} is harmonic in $\tilde{\omega} \cup \hat{\omega} \cup \bar{\omega}$.

The following is Theorem 1.1 in the case when $I = J = K = 1$.

Proposition 2.1. *For every large μ , there exists an $H_0^1(\Omega)$ weak solution u_μ of*

$$-\Delta u = (a^+ - \mu a^-)|u|^{p-2}u \quad \text{in } \Omega. \quad (4)$$

Furthermore, the family $\{u_\mu\}$ has the property that, modulo a subsequence,

$$u_\mu \rightharpoonup u \quad \text{in } H_0^1(\Omega) \text{ as } \mu \rightarrow +\infty, \quad (5)$$

where

$$u = \tilde{u} + \hat{u}, \quad (6)$$

$$\begin{cases} -\Delta \tilde{u} = a^+|\tilde{u}|^{p-2}\tilde{u} & \text{in } \tilde{\omega}, \\ \tilde{u}^\pm \not\equiv 0, \end{cases} \quad (7)$$

and

$$\begin{cases} -\Delta \hat{u} = a^+|\hat{u}|^{p-2}\hat{u} & \text{in } \hat{\omega}, \\ \hat{u}^+ \not\equiv 0, \hat{u}^- \equiv 0. \end{cases} \quad (8)$$

The solutions of (4) are the critical points of the C^2 functional $I_\mu : H_0^1(\Omega) \rightarrow \mathbb{R}$, defined by

$$I_\mu(u) = \frac{1}{2} \|u\|^2 - \frac{1}{p} \int (a^+ - \mu a^-) |u|^p.$$

We fix a function v such that $v = \tilde{v} + \hat{v}^+$, with $\tilde{v}^+, \tilde{v}^-, \hat{v} \not\equiv 0$ and

$$I'_\mu(v)(\tilde{v}^+) = I'_\mu(v)(\tilde{v}^-) = I'_\mu(v)(\hat{v}) = 0$$

for some (and hence all) $\mu > 0$.

The restriction of I_μ to $\underline{H}(\hat{\omega}) \oplus \underline{H}(\bar{\omega})$ is independent of μ and has a strict local minimum at zero. We fix a small $\rho_0 > 0$ such that zero is the unique minimizer of I_μ in $\{u \in \underline{H}(\hat{\omega}) \oplus \underline{H}(\bar{\omega}) : \max\{\|\hat{u}\|, \|\bar{u}\|\} \leq \rho_0\}$. For $0 < \rho \leq \rho_0$, we denote by c_ρ the positive constant

$$c_\rho := \inf_{\substack{u \in \underline{H}(\hat{\omega}) \oplus \underline{H}(\bar{\omega}) \\ \rho \leq \max\{\|\hat{u}\|, \|\bar{u}\|\} \leq \rho_0}} I_\mu(u). \quad (9)$$

The solutions of (4) will be obtained by minimizing the functional I_μ on the following Nehari-type set, \mathcal{N}_μ . Let ρ_0 be as above and $R > \|v\|$.

Definition 2.2. \mathcal{N}_μ is the set of functions $u = \tilde{u} + \hat{u} + \bar{u} + \underline{u} \in H_0^1(\Omega)$ such that

- (\mathcal{N}_i) $\tilde{u}^+, \tilde{u}^-, \hat{u}^+ \not\equiv 0$,
- (\mathcal{N}_{ii}) $I'_\mu(u)(\tilde{u}^+) = I'_\mu(u)(\tilde{u}^-) = I'_\mu(u)(\hat{u}^+) = 0$,
- (\mathcal{N}_{iii}) $I_\mu(u) \leq I_\mu(v) + 1$,
- (\mathcal{N}_{iv}) $\|\underline{u}\| \leq \min\{\|\tilde{u}^+\|, \|\tilde{u}^-\|, \|\hat{u}^+\|\} < \|\tilde{u} + \hat{u}^+\| \leq R$,
- (\mathcal{N}_v) $\max\{\|\hat{u}^-\|, \|\bar{u}\|\} \leq \rho_0$.

We remark that $v \in \mathcal{N}_\mu$ for all $\mu > 0$.

The square of the $H_0^1(\Omega)$ norm of u is equal to the sum of the squares of the $H_0^1(\Omega)$ norms of the components of u , but the p -th power of the $L^p(\Omega)$ norm of u does not have such a nice property. However, the next lemma says that this is almost the case when μ is large.

Lemma 2.3. Let $\delta > 0$ be given. There exists μ_δ such that, if $\mu > \mu_\delta$,

$$\forall u \in \mathcal{N}_\mu, \quad \int |\underline{u}|^p < \delta.$$

Proof. Suppose, by contradiction, that for some $\delta > 0$ there exists $\mu_n \rightarrow +\infty$ and $u_n \in \mathcal{N}_{\mu_n}$ with

$$\int |\underline{u}_n|^p \geq \delta. \quad (10)$$

As $\|u_n\|$ is bounded, we may suppose $u_n \rightharpoonup u$. We have $\underline{u}_n \rightharpoonup \underline{u}$ and $\underline{u} \equiv 0$ in $\Omega \setminus (\tilde{\omega} \cup \hat{\omega} \cup \bar{\omega})$. Otherwise, by (3) and modulo a subsequence,

$$\int a^- |\underline{u}_n|^p \geq c > 0.$$

This would contradict (\mathcal{N}_{iii}) for sufficiently large n :

$$\frac{1}{2} \|u_n\|^2 - \frac{1}{p} \int a^+ |u_n|^p + \frac{\mu_n}{p} \int a^- |\underline{u}_n|^p \leq I_\mu(v) + 1.$$

So the function \underline{u} belongs to $\underline{H}(\tilde{\omega}) \oplus \underline{H}(\hat{\omega}) \oplus \underline{H}(\bar{\omega})$ and is harmonic in $\tilde{\omega} \cup \hat{\omega} \cup \bar{\omega}$. It follows that \underline{u} must be identically equal to zero in Ω . This contradicts (10). \square

Usually one may obtain a lower bound for the $H_0^1(\Omega)$ norm of \tilde{u}^+ , \tilde{u}^- and \hat{u}^+ from (\mathcal{N}_i) and a condition like (\mathcal{N}_{ii}) . Here, in addition, we require the first inequality in (\mathcal{N}_{iv}) to prove

Lemma 2.4. *There exists a constant κ , independent of μ , such that*

$$\forall u \in \mathcal{N}_\mu, \quad \min \{ \|\tilde{u}^+\|, \|\tilde{u}^-\|, \|\hat{u}^+\| \} \geq \kappa > 0. \quad (11)$$

Proof. Let w be one of the three functions \tilde{u}^+ , $-\tilde{u}^-$ or \hat{u}^+ . Denote by χ the characteristic function of the set $\{x \in \Omega : w(x) \neq 0\}$ and let c be the Sobolev constant $(\int |v|^p)^{1/p} \leq c \|v\|$, $\forall v \in H_0^1(\Omega)$. From $I'_\mu(u)w = 0$,

$$\begin{aligned} \|w\|^2 &= \int a^+ |u|^{p-2} u w \leq \|a\|_\infty \left(\int \chi |u|^p \right)^{\frac{p-1}{p}} \left(\int |w|^p \right)^{\frac{1}{p}} \\ &\leq \|a\|_\infty c^p (\|\underline{u}\| + \|w\|)^{p-1} \|w\| \leq 2^{p-1} \|a\|_\infty c^p \|w\|^p, \end{aligned}$$

because of the first inequality in (\mathcal{N}_{iv}) . Since $w \neq 0$, due to (\mathcal{N}_i) , we may take

$$\kappa = (2^{p-1} \|a\|_\infty c^p)^{-1/(p-2)}.$$

\square

Now we fix a μ and turn to minimizing sequences (u_n) for I_μ restricted to \mathcal{N}_μ . Later it will be important that the limit of such a sequence has a neighborhood whose points satisfy (\mathcal{N}_i) , (\mathcal{N}_{iii}) , (\mathcal{N}_{iv}) and (\mathcal{N}_v) . This follows from

Lemma 2.5. *Let \overline{R} be fixed, $\|v\| < \overline{R} < R$, and δ be given, $0 < \delta < \rho_0$. There exists $\mu_\delta > 0$ such that for every $\mu > \mu_\delta$ and every minimizing sequence (u_n) for I_μ restricted to \mathcal{N}_μ , we have, for large n ,*

- (a) $I_\mu(u_n) \leq I_\mu(v) + \frac{1}{2}$,
- (b) $\|\tilde{u}_n + \hat{u}_n^+\| < \overline{R}$,
- (c) $\max\{\|\hat{u}_n^-\|, \|\bar{u}_n\|\} < \delta$,
- (d) $\|\underline{u}_n\| < \delta$;

also

- (e) $\frac{\mu}{p} \int a^- |\underline{u}_n|^p < \delta$.

Proof. (a) Immediate since (u_n) is minimizing and $v \in \mathcal{N}_\mu$ for all μ .

(b) Suppose

$$\|\tilde{u}_n + \hat{u}_n^+\| \geq \overline{R} \quad (12)$$

for large n .

$$\begin{aligned} I_\mu(u_n) &= \frac{1}{2} \|\tilde{u}_n + \hat{u}_n^+\|^2 + \frac{1}{2} \|\hat{u}_n^-\|^2 + \frac{1}{2} \|\bar{u}_n\|^2 + \frac{1}{2} \|\underline{u}_n\|^2 \\ &\quad - \frac{1}{p} \int a^+ |u_n|^{p-2} u_n (\tilde{u}_n + \hat{u}_n^+) + \frac{1}{p} \int a^+ |u_n|^{p-2} u_n \hat{u}_n^- \\ &\quad - \frac{1}{p} \int a^+ |u_n|^{p-2} u_n \bar{u}_n - \frac{1}{p} \int a^+ |u_n|^{p-2} u_n \underline{u}_n + \frac{\mu}{p} \int a^- |\underline{u}_n|^p \\ &\geq \left(\frac{1}{2} - \frac{1}{p} \right) \|\tilde{u}_n + \hat{u}_n^+\|^2 + o(1). \end{aligned}$$

Here and henceforth $o(1)$ denotes a value, independent of $u \in \mathcal{N}_\mu$, that can be made arbitrarily small by choosing μ sufficiently large. For the proof of the last inequality we used (\mathcal{N}_{ii}) ,

$$\frac{1}{2} \|\hat{u}_n^-\|^2 + \frac{1}{p} \int a^+ |u_n|^{p-2} u_n \hat{u}_n^- \geq o(1)$$

and

$$\frac{1}{2} \|\bar{u}_n\|^2 - \frac{1}{p} \int a^+ |u_n|^{p-2} u_n \bar{u}_n \geq o(1)$$

(consequences of (\mathcal{N}_v) and Lemma 2.3),

$$-\frac{1}{p} \int a^+ |u_n|^{p-2} u_n \underline{u}_n = o(1)$$

(consequence of (\mathcal{N}_{iv}) , (\mathcal{N}_v) and Lemma 2.3), and

$$\frac{1}{2} \|\underline{u}_n\|^2 + \frac{\mu}{p} \int a^- |\underline{u}_n|^p \geq 0.$$

We now use (12) and the definition of \overline{R} . For sufficiently large μ ,

$$I_\mu(u_n) \geq \left(\frac{1}{2} - \frac{1}{p}\right) \overline{R}^2 + o(1) > \left(\frac{1}{2} - \frac{1}{p}\right) \|v\|^2 + c = I_\mu(v) + c,$$

for some $c > 0$. This contradicts the fact that (u_n) is minimizing.

(c) Suppose $\|\hat{u}_n^-\| \geq \delta$ for large n . As in (b), we have

$$\begin{aligned} I_\mu(u_n) &= I_\mu(u_n + \hat{u}_n^-) + \frac{1}{2} \|\hat{u}_n^-\|^2 - \frac{1}{p} \int a^- |\hat{u}_n^-|^p + o(1) \\ &\geq I_\mu(u_n + \hat{u}_n^-) + c_\delta + o(1), \end{aligned}$$

due to Lemma 2.3 and then (9). This implies that

$$\lim I_\mu(u_n) > \liminf I_\mu(u_n + \hat{u}_n^-),$$

for sufficiently large μ , and contradicts the assumption that (u_n) is minimizing, because $u_n + \hat{u}_n^- \in \mathcal{N}_\mu$. Similarly, one proves that $\|\bar{u}_n\| \geq \delta$ for large n leads to a contradiction, for sufficiently large μ , because $u_n - \bar{u}_n \in \mathcal{N}_\mu$.

(d) Suppose $\|\underline{u}_n\| \geq \delta$ for large n . From (\mathcal{N}_{ii}) and Lemma 2.3, we know

$$\begin{aligned} \|\tilde{u}_n^+\|^2 &= \int a^+ |\tilde{u}_n^+|^p + o(1), \\ \|\tilde{u}_n^-\|^2 &= \int a^+ |\tilde{u}_n^-|^p + o(1), \\ \|\hat{u}_n^+\|^2 &= \int a^+ |\hat{u}_n^+|^p + o(1). \end{aligned}$$

We define \tilde{r}_n , \tilde{s}_n and \hat{t}_n by

$$\tilde{r}_n = \left(\frac{\|\tilde{u}_n^+\|^2}{\int a^+ |\tilde{u}_n^+|^p} \right)^{\frac{1}{p-2}}, \quad \tilde{s}_n = \left(\frac{\|\tilde{u}_n^-\|^2}{\int a^+ |\tilde{u}_n^-|^p} \right)^{\frac{1}{p-2}}, \quad \hat{t}_n = \left(\frac{\|\hat{u}_n^+\|^2}{\int a^+ |\hat{u}_n^+|^p} \right)^{\frac{1}{p-2}},$$

so that $\tilde{r}_n, \tilde{s}_n, \hat{t}_n = 1 + o(1)$ by Lemma 2.4, and

$$v_n := \tilde{r}_n \tilde{u}_n^+ - \tilde{s}_n \tilde{u}_n^- + \hat{t}_n \hat{u}_n^+ - \hat{u}_n^- + \bar{u}_n.$$

Provided μ is large, we can guarantee $v_n \in \mathcal{N}_\mu$ for large n due to (a), (b), (c) and Lemma 2.4. We now obtain an upper bound for $I_\mu(v_n)$:

$$\begin{aligned}
I_\mu(v_n) &= I_\mu(\tilde{u}_n + \hat{u}_n + \bar{u}_n) + o(1) \\
&\leq I_\mu(u_n) + o(1) \\
&\quad - \left(\frac{1}{2} \|\underline{u}_n\|^2 - \frac{1}{p} \int a^+ (|u_n|^p - |u_n - \underline{u}_n|^p) + \frac{\mu}{p} \int a^- |\underline{u}_n|^p \right) \\
&\leq I_\mu(u_n) + o(1) - \frac{1}{2} \|\underline{u}_n\|^2 \\
&\leq I_\mu(u_n) + o(1) - \frac{1}{2} \delta^2.
\end{aligned} \tag{13}$$

This implies that $\liminf I_\mu(v_n) < \lim I_\mu(u_n)$ for sufficiently large μ , which is impossible.

(e) Follows from inequality (13). \square

3 Existence of a minimizer in \mathcal{N}_μ

For each $u \in \mathcal{N}_\mu$, we consider the 3-dimensional manifold with boundary in $H_0^1(\Omega)$ parametrized on $[0, 2]^3$ by

$$\varsigma(\tilde{r}, \tilde{s}, \hat{t}) = \tilde{r}\tilde{u}^+ - \tilde{s}\tilde{u}^- + \hat{t}\hat{u}^+ - \hat{u}^- + \bar{u} + \underline{u}. \tag{14}$$

We call f the function $I_\mu \circ \varsigma$, so that

$$\begin{aligned}
f(\tilde{r}, \tilde{s}, \hat{t}) &= \frac{\tilde{r}^2}{2} \|\tilde{u}^+\|^2 + \frac{\tilde{s}^2}{2} \|\tilde{u}^-\|^2 + \frac{\hat{t}^2}{2} \|\hat{u}^+\|^2 + K \\
&\quad - \frac{1}{p} \int a^+ |\tilde{r}\tilde{u}^+ + \underline{u}|^p - \frac{1}{p} \int a^+ |\underline{u} - \tilde{s}\tilde{u}^-|^p - \frac{1}{p} \int a^+ |\hat{t}\hat{u}^+ + \underline{u}|^p,
\end{aligned}$$

with

$$\begin{aligned}
K &= \frac{1}{2} \|\hat{u}^-\|^2 + \frac{1}{2} \|\bar{u}\|^2 + \frac{1}{2} \|\underline{u}\|^2 \\
&\quad - \frac{1}{p} \int a^+ |\underline{u} - \hat{u}^-|^p - \frac{1}{p} \int a^+ |\bar{u} + \underline{u}|^p + \frac{\mu}{p} \int a^- |\underline{u}|^p.
\end{aligned}$$

Two properties of f are immediate, namely $f(1, 1, 1) = I_\mu(u)$ and $\nabla f(1, 1, 1) = 0$ by (\mathcal{N}_{ii}) . The critical point $(1, 1, 1)$ is characterized in

Lemma 3.1. *For μ sufficiently large, independent of $u \in \mathcal{N}_\mu$, the point $(1, 1, 1)$ is an absolute maximum of f . Furthermore, if*

$$|(\tilde{r}, \tilde{s}, \hat{t}) - (1, 1, 1)| \geq \theta > 0,$$

then

$$f(\tilde{r}, \tilde{s}, \hat{t}) \leq f(1, 1, 1) - d_\theta. \quad (15)$$

The constant $d_\theta > 0$ may be chosen independent of u and μ .

Proof. We define an auxiliary function $g : [0, 2]^3 \rightarrow \mathbb{R}$ by

$$\begin{aligned} g(\tilde{r}, \tilde{s}, \hat{t}) &:= \left(\frac{\tilde{r}^2}{2} - \frac{\tilde{r}^p}{p} \right) \|\tilde{u}^+\|^2 + \left(\frac{\tilde{s}^2}{2} - \frac{\tilde{s}^p}{p} \right) \|\tilde{u}^-\|^2 \\ &\quad + \left(\frac{\hat{t}^2}{2} - \frac{\hat{t}^p}{p} \right) \|\hat{u}^+\|^2 + K, \end{aligned}$$

which satisfies $\nabla g(1, 1, 1) = 0$ and

$$D^2 g(1, 1, 1) = -(p-2) \operatorname{diag} \left\{ \|\tilde{u}^+\|^2, \|\tilde{u}^-\|^2, \|\hat{u}^+\|^2 \right\} \leq -(p-2)\kappa I,$$

where κ was defined in Lemma 2.4. One easily checks that in a small neighborhood of $(1, 1, 1)$ the second derivative $D^2 g$ is below a negative definite matrix which is independent of $u \in \mathcal{N}_\mu$. We also have that, for any derivative D^α with $|\alpha| \leq 2$,

$$|D^\alpha f - D^\alpha g| = o(1), \quad (16)$$

by Lemma 2.3; notice that the right-hand-side is uniform in u and μ . Thus, by (16) with $|\alpha| = 2$, f has a strict local maximum at $(1, 1, 1)$. We take $\alpha = 0$ to conclude this maximum is absolute. Of course, the previous two statements hold provided μ is sufficiently large. \square

Let μ be fixed and (u_n) be a minimizing sequence for I_μ restricted to \mathcal{N}_μ . Since \mathcal{N}_μ is bounded in $H_0^1(\Omega)$, we may assume

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega).$$

Lemma 3.2. *If μ is sufficiently large, the function u belongs to \mathcal{N}_μ . Therefore (by the lower semi-continuity of the norm) the function u is a minimizer of I_μ restricted to \mathcal{N}_μ .*

Proof. We may assume $\tilde{u}_n^+ \rightharpoonup \tilde{u}^+$, $\tilde{u}_n^- \rightharpoonup \tilde{u}^-$, $\hat{u}_n^+ \rightharpoonup \hat{u}^+$ in $H_0^1(\Omega)$, since $w_n \rightharpoonup w$ in $H_0^1(\Omega)$ implies a subsequence of w_n converges pointwise a.e. to w . From (\mathcal{N}_{ii}) and Lemma 2.4,

$$\min \left\{ \int a^+ |u|^{p-2} u \tilde{u}^+, - \int a^+ |u|^{p-2} u \tilde{u}^-, \int a^+ |u|^{p-2} u \hat{u}^+ \right\} \geq \kappa.$$

These three integrals are also bounded above by a constant independent of μ because \mathcal{N}_μ is bounded. It follows from Lemma 2.3 that the integrals

$$\int a^+ |\tilde{u}^+|^p, \quad \int a^+ |\tilde{u}^-|^p, \quad \int a^+ |\hat{u}^+|^p$$

are bounded below by a positive constant independent of μ . The Sobolev inequality now implies that the norms

$$\|\tilde{u}^+\|, \quad \|\tilde{u}^-\|, \quad \|\hat{u}^+\|$$

are bounded below by a positive constant independent of μ . From the lower-semicontinuity of the norm,

$$\|\tilde{u}^+\| \leq \liminf \|\tilde{u}_n^+\|, \quad \|\tilde{u}^-\| \leq \liminf \|\tilde{u}_n^-\|, \quad \|\hat{u}^+\| \leq \liminf \|\hat{u}_n^+\|. \quad (17)$$

We wish to prove that equalities hold. Otherwise, choose $(\tilde{r}, \tilde{s}, \hat{t})$, defined by

$$\begin{aligned} \tilde{r} &= \left(\frac{\|\tilde{u}^+\|^2}{\int a^+ |u|^{p-2} u \tilde{u}^+} \right)^{\frac{1}{p-2}}, \quad \tilde{s} = \left(\frac{\|\tilde{u}^-\|^2}{-\int a^+ |u|^{p-2} u \tilde{u}^-} \right)^{\frac{1}{p-2}}, \\ \hat{t} &= \left(\frac{\|\hat{u}^+\|^2}{\int a^+ |u|^{p-2} u \hat{u}^+} \right)^{\frac{1}{p-2}}, \end{aligned}$$

so that the function

$$w := \tilde{r}\tilde{u}^+ - \tilde{s}\tilde{u}^- + \hat{t}\hat{u}^+ - \hat{u}^- + \bar{u} + \underline{u}$$

satisfies (\mathcal{N}_{ii}) . By (17), the strong convergence in $L^p(\Omega)$, and what we have just seen,

$$(\tilde{r}, \tilde{s}, \hat{t}) \in [c, 1]^3 \setminus \{(1, 1, 1)\},$$

for some $c > 0$ independent of μ . The function w clearly satisfies (\mathcal{N}_i) and (\mathcal{N}_v) . Lemma 2.3 guarantees that (\mathcal{N}_{iv}) is satisfied for sufficiently large μ . Consider the estimate

$$\begin{aligned} I_\mu(\tilde{r}\tilde{u}^+ - \tilde{s}\tilde{u}^- + \hat{t}\hat{u}^+ - \hat{u}^- + \bar{u} + \underline{u}) \\ &< \liminf I_\mu(\tilde{r}\tilde{u}_n^+ - \tilde{s}\tilde{u}_n^- + \hat{t}\hat{u}_n^+ - \hat{u}_n^- + \bar{u}_n + \underline{u}_n) \\ &\leq \lim I_\mu(u_n), \end{aligned}$$

where the last inequality is due to Lemma 3.1. It shows that w satisfies (\mathcal{N}_{iii}) . Therefore $w \in \mathcal{N}_\mu$ and $I_\mu(w) < \lim I_\mu(u_n)$. This is a contradiction. We have established that equality holds in all three of (17). Therefore $u \in \mathcal{N}_\mu$ for large μ . \square

4 A minimizer in \mathcal{N}_μ is a critical point

In the previous section we obtained a minimizer u of I_μ on \mathcal{N}_μ . We will now prove that this minimizer is indeed a critical point of I_μ . This will be done by using a deformation argument on the manifold introduced above. Let σ be the restriction to the interval $[1/2, 2]^3$ of the ς corresponding to the minimizer u . Recall ς was defined in (14). We define a negative gradient flow in a neighborhood of u in the following way. Let $B_\rho(u) := \{w \in H_0^1(\Omega) : \|w - u\| < \rho\}$, where ρ is chosen small enough so that

$$\sigma(\tilde{r}, \tilde{s}, \hat{t}) \in B_\rho(u) \Rightarrow \frac{1}{2} < \tilde{r}, \tilde{s}, \hat{t} < 2 \quad (18)$$

and $w \in B_\rho(u)$ implies that w satisfies (\mathcal{N}_i) , (\mathcal{N}_{iii}) , (\mathcal{N}_{iv}) and (\mathcal{N}_v) , for sufficiently large μ . Such a ρ exists because the function u satisfies (11) and (a), (b), (c) and (d) of Lemma 2.5. Let φ be a Lipschitz function, $\varphi : H_0^1(\Omega) \rightarrow [0, 1]$, such that $\varphi = 1$ on $B_{\rho/2}(u)$ and $\varphi = 0$ on the complement of $B_\rho(u)$. Consider the Cauchy problem

$$\begin{cases} \frac{d\eta}{d\tau} = -\varphi(\eta)\nabla I_\mu(\eta), \\ \eta(0) = w, \end{cases} \quad (19)$$

whose solution we denote by $\eta(\tau; w)$. For $\tau \geq 0$, let

$$\sigma_\tau(\tilde{r}, \tilde{s}, \hat{t}) = \eta(\tau; \sigma(\tilde{r}, \tilde{s}, \hat{t})).$$

Lemma 4.1. *The set $\sigma_\tau([1/2, 2]^3)$ intersects \mathcal{N}_μ in a nonempty set.*

Proof. Consider the maps $\tilde{\phi}^\pm, \hat{\phi}, \tilde{\psi}^\pm, \hat{\psi}$ from $\{w \in H_0^1(\Omega) : \tilde{w}^\pm \neq 0, \hat{w}^+ \neq 0\}$ to \mathbb{R} , defined by

$$\begin{aligned} \tilde{\phi}^\pm(w) &= \frac{\pm \int a^+ |w|^{p-2} w \tilde{w}^\pm}{\|\tilde{w}^\pm\|^2}, & \hat{\phi}(w) &= \frac{\int a^+ |w|^{p-2} w \hat{w}^+}{\|\hat{w}^+\|^2}, \\ \tilde{\psi}^\pm(w) &= \frac{\int a^+ |\tilde{w}^\pm|^p}{\|\tilde{w}^\pm\|^2}, & \hat{\psi}(w) &= \frac{\int a^+ |\hat{w}^+|^p}{\|\hat{w}^+\|^2}. \end{aligned}$$

These maps are well defined on $\sigma_\tau([1/2, 2]^3)$, because if $w \in B_\rho(u)$, then w satisfies (\mathcal{N}_i) . We finally define

$$\Phi_\tau := (\tilde{\phi}^+, \tilde{\phi}^-, \hat{\phi}) \circ \sigma_\tau$$

and

$$\Psi := (\tilde{\psi}^+, \tilde{\psi}^-, \hat{\psi}) \circ \sigma,$$

from $([1/2, 2]^3)$ to \mathbb{R}^3 . Since $\int |\underline{u}|^p = o(1)$ uniformly in u and μ and the value of κ in Lemma 2.4 is independent of μ ,

$$\begin{aligned}\Psi(\tilde{r}, \tilde{s}, \hat{t}) &= \left(\tilde{r}^{p-2} \tilde{\psi}^+(u), \tilde{s}^{p-2} \tilde{\psi}^-(u), \hat{t}^{p-2} \hat{\psi}(u) \right) \\ &= \left((1 + o(1)) \tilde{r}^{p-2}, (1 + o(1)) \tilde{s}^{p-2}, (1 + o(1)) \hat{t}^{p-2} \right),\end{aligned}\quad (20)$$

with the last three $o(1)$ independent of u and μ . As a consequence,

$$\text{dist} \left(\Psi \left(\partial[1/2, 2]^3 \right), (1, 1, 1) \right) \geq c > 0,$$

the constant c being independent of u and μ . We deduce from (20) that for large μ ,

$$\deg \left(\Psi, [1/2, 2]^3, (1, 1, 1) \right) = 1.$$

Notice that condition (18) and the definition of the flux (19) guarantee

$$\Phi_\tau|_{\partial[1/2, 2]^3} = \Phi_0|_{\partial[1/2, 2]^3} = \Psi|_{\partial[1/2, 2]^3} + o(1)$$

and therefore

$$\deg \left(\Phi_\tau, [1/2, 2]^3, (1, 1, 1) \right) = 1.$$

for μ large enough. This proves that

$$\sigma_\tau \left([1/2, 2]^3 \right) \cap \mathcal{N}_\mu \neq \emptyset.$$

□

We are ready to give the

Proof of Proposition 2.1. Let μ be large and u_μ be a minimizer of I_μ restricted to \mathcal{N}_μ . The existence of such a u_μ was proven in Lemma 3.2. Suppose that $I'_\mu(u_\mu) \neq 0$. By Lemma 3.1, with $u = u_\mu$, $\max I_\mu \circ \sigma \left([1/2, 2]^3 \right) = I_\mu(u_\mu)$, and so for any small $\tau > 0$,

$$\max I_\mu \circ \sigma_\tau \left([1/2, 2]^3 \right) < I_\mu(u_\mu).$$

This contradicts Lemma 4.1. So $I'_\mu(u_\mu) = 0$, and the minimizer of I_μ on \mathcal{N}_μ is a weak solution of (4).

Consider now u as in (5). Properties (6), (7) and (8) follow from Lemma 2.4 and Lemma 2.5 (c), (d), as

$$\min \left\{ \int a^+ |u_\mu|^{p-2} u_\mu \tilde{u}_\mu^+, - \int a^+ |u_\mu|^{p-2} u_\mu \tilde{u}_\mu^-, \int a^+ |u_\mu|^{p-2} u_\mu \hat{u}_\mu^+ \right\} \geq \kappa.$$

□

Theorem 1.1 can be proved as Proposition 2.1 with obvious adaptations.

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